

## ON FIELDS WITH ONLY FINITELY MANY MAXIMAL SUBRINGS

ALBORZ AZARANG

Department of Mathematics, Shahid Chamran University of Ahvaz, Ahvaz, Iran  
a\_azarang@scu.ac.ir

**ABSTRACT.** Fields with only finitely many maximal subrings are completely determined. We show that such fields are certain absolutely algebraic fields and give some characterization of them. In particular, we show that the following conditions are equivalent for a field  $E$ :

- (1)  $E$  has only finitely many maximal subrings.
- (2)  $E$  has a subfield  $F$  which has no maximal subrings and  $[E : F]$  is finite.
- (3) Every descending chain  $\cdots \subset R_2 \subset R_1 \subset R_0 = E$  where each  $R_i$  is a maximal subring of  $R_{i-1}$ ,  $i \geq 1$ , is finite.

Moreover, if one of the above equivalent conditions holds, then  $F$  is unique and contains all subfields of  $E$  which have no maximal subrings. Furthermore, all chains in (3) have the same length,  $m$  say, and  $R_m = F$ , where  $m$  is the sum of all powers of primes in the factorization of  $[E : F]$  into prime numbers. We also determine when certain affine rings have only finitely many maximal subrings. In particular, we prove that if  $R = F[\alpha_1, \dots, \alpha_n]$  is an affine integral domain over a field  $F$ , then  $R$  has only finitely many maximal subrings if and only if  $F$  has only finitely many maximal subrings and each  $\alpha_i$  is algebraic over  $F$ , which is similar to the celebrated Zariski's Lemma. Finally, we show that if  $R$  is an uncountable PID then  $R$  has at least  $|R|$ -many maximal subrings.

## INTRODUCTION

All rings in this note are commutative with  $1 \neq 0$ . All subrings, ring extensions, homomorphisms and modules are unital. A proper subring  $S$  of a ring  $R$  is called a maximal subring if  $S$  is maximal with respect to inclusion in the set of all proper subrings of  $R$ . Not every ring possesses maximal subrings (for example the algebraic closure of a finite field has no maximal subrings, see [14, Corollary 2.7] or [7, Remark 1.13]; also see [6, Example 2.6] and [9, Example 3.19] for more examples of rings which have no maximal subrings). A ring which possesses a maximal subring is said to be submaximal, see [3], [7] and [9]. If  $S$  is a maximal subring of a ring  $R$ , then the extension  $S \subseteq R$  is called a minimal ring extension (see [21]) or an adjacent extension too (see [16]).

In [29], M. L. Modica, a student of Kaplansky, studied maximal subrings of affine integral domains. Let us recall some result from it. Assume that  $K$  is an algebraically closed field and  $T$  be an affine integral domain with the quotient field  $L$ . Then in [29], it is shown that the maximal subrings of  $T$  which contain  $K$  and  $T$  is integral over them are affine over  $K$  (by Artin-Tate Theorem) and are of the form  $R = K + I$ , where  $I = (R : T) \in \text{Max}(R)$ ,  $I \notin \text{Max}(T)$ . Moreover, either there exist exactly two maximal ideals  $M$  and  $N$  of  $T$  such that  $I = R \cap M = R \cap N$  and therefore  $I = M \cap N$  (hence  $|\text{Max}(T/I)| = 2$ , and note that in algebraic geometry this mean that if  $P$  and  $Q$  are two points such that  $M_P = M$  and  $M_Q = N$ , then  $R = K + (M_P \cap M_Q) = \{f \in T \mid f(P) = f(Q)\}$ ), or there exists exactly one maximal ideals  $M$  of  $T$  which contains  $I$  and  $T/I \cong K[t]/(t^2)$  (therefore  $M^2 \subseteq I$ ), see [29, Theorem 3]. Next, it is shown that (for arbitrary field  $K$  not necessary algebraically closed) if  $R$  is integrally closed in  $T$  and  $\dim(T) = \text{tr.deg}(T/K) = m \geq 2$ , then  $R$  is not affine over  $K$  (see [29, Theorem 16]). It is observed that if  $m = 1$  (therefore  $L/K$  is a function field of one variable), then there exist only finitely many DVRs of  $L$  containing  $K$  but not contain  $T$ , namely  $W_1, \dots, W_r$  (see [29, Lemma 7]). If  $r = 1$ , then  $T$  has no maximal subring containing  $K$  which is integrally closed in  $T$ . But, if  $r \geq 2$ , then  $T$  has exactly  $r$  distinct maximal subrings which are integrally closed in  $T$  and contain  $K$ , namely  $T \cap W_1, \dots, T \cap W_r$  and all of them are affine over  $K$ , see [29, Theorem 15].

2000 *Mathematics Subject Classification.* 13B99, 13A15, 13C13, 13G05, 13E05, 13C99.

*Key words and phrases.* Fields, Maximal subring, field generated set, set of maximal subrings, chain of maximal subrings, affine domain.

Next, we also recall some fact about finiteness conditions on the set of subrings, intermediate rings of ring extensions and overrings of an integral domains which are closely related to our study. In [32], it is shown that if a ring (possibly noncommutative) satisfying both ascending and descending chain conditions on its subrings then the ring must be finite. Gilmer studied integral domains with some finiteness conditions on the set of overrings, see [23]. But it seems the chain conditions and finiteness conditions on the set of intermediate rings of a ring extension  $R \subseteq T$  was first studied in [1], for general commutative ring extensions, in order to generalize the Steinitz's Primitive Element Theorem for field extensions (see [15, Theorem 7.9.3]). A ring extension  $R \subseteq T$  with only finitely many intermediate rings is called *FIP*-extension in [1]. *FIP*-extensions are also studied in [18] and recently *FIP*-extensions are characterized in [20]. In [31], Rosenfeld proved that a (possibly noncommutative) unital ring with only finitely many subrings (not necessarily unital) is finite. Bell and Gilmer have given elementary proofs of this result; see [13] and [22], respectively. Recently, Dobbs et al., studied commutative unital rings with only finitely many unital subrings. They characterized such rings first in [17] for singly generated unital rings and later in [19] for general commutative rings. In [12, 25, 27, 28], it is proved that if a ring  $R$  has a finite maximal subrings, then  $R$  is finite. Korobkov characterized finite rings with exactly two maximal subrings, see [26].

The existence of maximal subrings of commutative rings were first studied in [2, 3], [5-10] and more recently in [11]. In this article, we are interested in characterizing fields with only finitely many maximal subrings. Moreover, we also settle the question that: when do affine integral domains over fields have only finitely many maximal subrings?

Next, let us recall some standard definitions and notation from commutative ring theory which will be used throughout the paper, see [24]. An integral domain  $D$  is called *G*-domain if the quotient field of  $D$  is finitely generated as a  $D$ -algebra. A prime ideal  $P$  of a ring  $R$  is called *G*-ideal if  $R/P$  is a *G*-domain. A ring  $R$  is called Hilbert if every *G*-ideal of  $R$  is maximal. As usual, let  $Char(R)$ ,  $U(R)$ ,  $N(R)$ ,  $J(R)$ ,  $Max(R)$ ,  $Spec(R)$  and  $Min(R)$ , denote the characteristic, the set of all units, the nil radical ideal, the Jacobson radical ideal, the set of all maximal ideals, the set of all prime ideals and the set of all minimal prime ideals of a ring  $R$ , respectively. We also call a ring  $R$ , not necessarily noetherian, is semilocal (resp. local) if  $Max(R)$  is finite (resp.  $|Max(R)| = 1$ ). For any ring  $R$ , let  $Z = \mathbb{Z} \cdot 1_R = \{n \cdot 1_R \mid n \in \mathbb{Z}\}$ , be the prime subring of  $R$ . We denote the finite field with  $p^n$  elements, where  $p$  is prime and  $n \in \mathbb{N}$ , by  $F_{p^n}$ . Fields which are algebraic over  $F_p$  for some prime number  $p$ , are called absolutely algebraic field. If  $D$  is an integral domain, then we denote the set of all non-associate irreducible elements of  $D$  by  $Irr(D)$ . Also, we denote the set of all natural prime numbers by  $\mathbb{P}$ . Suppose that  $D \subseteq R$  is an extension of domains, then by Zorn's Lemma, there exists a maximal subset  $X$  of  $R$  which is algebraically independent over  $D$ . Clearly  $R$  is algebraic over  $D[X]$ . If  $E$  and  $F$  are the quotient fields of  $D$  and  $R$ , respectively, then  $X$  can be shown to be a transcendence basis for  $F/E$ . The transcendence degree of  $F$  over  $E$  is the cardinality of a transcendence basis for  $F/E$ . We denote the transcendence degree of  $F$  over  $E$  by  $tr.deg(F/E)$ . If  $R$  is a ring, then  $RgMax(R)$  denotes the set of all maximal subrings of  $R$ .

Now, let us sketch a brief outline of this paper. Section 1, contains some preliminaries and definitions from [7], [9] and [3]. In this section, we characterize the set of all maximal subrings of absolutely algebraic fields. Consequently we show that an absolutely algebraic field  $E$  has only finitely many maximal subrings if and only if  $E = \bigcup_{n \in T} F_{q^n}$ , where  $q$  is a prime number and  $T$  consists of 1 and certain natural numbers. We also determine absolutely algebraic fields  $E$  for which every proper subfield of  $E$  can be embedded in a maximal subring of  $E$ . In section 2, we give some characterizations of fields  $E$  for which  $RgMax(E)$  is finite. In particular, we show that the following conditions are equivalent for a field  $E$ :

- (1)  $E$  has only finitely many maximal subrings.
- (2)  $E$  has a subfield  $F$  which has no maximal subring and  $[E : F]$  is finite.
- (3) every descending chain  $\cdots \subset R_2 \subset R_1 \subset R_0 = E$ , where each  $R_i$  is a maximal subring of  $R_{i-1}$  for  $i \geq 1$ , is finite.

Moreover, if one of the above conditions holds then  $F$  is unique and all chains in (3) have the same length,  $m$  say, and  $R_m = F$ . Furthermore, we show that, if  $\mathcal{E}$  is the set of all fields, up to isomorphism, which have only finitely many maximal subrings, then  $|\mathcal{E}| = 2^{\aleph_0}$ .

Finally, in Section 3, we study certain affine rings with only finitely many maximal subrings. We prove that if  $F$  is a field and  $R = F[\alpha_1, \dots, \alpha_n]$  is an affine integral domain, then  $R$  has only finitely many maximal subrings if and only if  $F$  has only finitely many maximal subrings and each  $\alpha_i$  is algebraic over

$F$  (this result resemble the Zariski's Lemma, which say that  $R$  is a field (or semilocal) if and only if each  $\alpha_i$  is algebraic over  $F$ ). We show that if  $R$  is a ring and  $x$  is an indeterminate over  $R$ , then there exists an infinite chain  $\cdots \subset R_1 \subset R_0 = R[x]$ , where each  $R_i$  is a maximal subring of  $R_{i-1}$  and  $R[x]$  is integral over each  $R_i$ , for  $i \geq 1$ . Next, we show that if  $R \subseteq T$  is an affine extension of rings and  $T$  has only finitely many maximal subrings, then  $R$  is zero-dimensional (resp. semilocal) if and only if  $T$  is zero-dimensional (resp. semilocal); consequently, we prove that  $R$  is artinian if and only if  $T$  is artinian. In the other main theorem of this section we characterize exactly the maximal subrings of  $K[x]/(x^2)$ , where  $K$  is a field. In particular, we prove that for a field  $K$ , the ring  $K[x]/(x^2)$  has only finitely many maximal subrings if and only if  $K$  has only finitely many maximal subrings; and in this case  $|RgMax(K[x]/(x^2))| = 1 + |RgMax(K)|$ . Finally in this section, we prove that if  $R$  is an uncountable PID, then  $|RgMax(R)| \geq |R|$ .

## 1. PRELIMINARIES AND $FG$ -SETS

We begin this section with the following facts.

**Theorem 1.1.** [3, Theorem 1.2]. *Let  $R$  be a ring and  $D$  be a subring of  $R$  which is a UFD. If there exists an irreducible element  $p \in D$  such that  $\frac{1}{p} \in R$ , then  $R$  has a maximal subring  $S$  which is integrally closed in  $R$  and  $\frac{1}{p} \notin S$ .*

**Corollary 1.2.** *Let  $R$  be a ring. Then the following statements hold:*

- (1) [3, Corollary 1.5]. *If  $R$  has zero characteristic and there exists a natural number  $n > 1$  such that  $\frac{1}{n} \in R$ , then  $R$  is submaximal.*
- (2) [9, Theorem 2.4]. *If  $R$  has a unit element  $x$  which is not algebraic over  $Z$ , then  $R$  is submaximal.*
- (3) [3, Proposition 1.18] or [9, Corollary 2.6]. *Either  $R$  is submaximal or  $J(R)$  is algebraic over  $Z$ .*
- (4) [3, Corollary 1.19]. *Let  $R$  be an integral domain with  $J(R) \neq 0$ . Then any  $R$ -algebra is submaximal. In particular, any algebra over a non-field  $G$ -domain is submaximal.*

In order to characterize  $RgMax(E)$  for the subfields  $E$  of  $\bar{F}_p$  we borrow the following definition from [7].

**Definition 1.3.** [7, Definition 1.5]. Let  $\mathbb{N}$  be the set of positive integers and  $T \subseteq \mathbb{N}$ . Then  $T$  is said to be a field generating set (briefly  $FG$ -set) if  $E = \bigcup_{n \in T} F_{p^n}$  is a subfield of  $\bar{F}_p$ , where  $p$  is a prime number; and  $T$  must be such that if  $T \subseteq T' \subseteq \mathbb{N}$  and  $E = \bigcup_{n \in T'} F_{p^n}$ , then  $T = T'$ .

*Remark 1.4.* [7, Remark 1.7]. One can easily see that there is a one-one order preserving correspondence between the  $FG$ -subsets of  $\mathbb{N}$  and the subfields of  $\bar{F}_p$ , see also the Steinitz's numbers and their properties in either [14] or [30]. Hence if  $E$  is a subfield of  $\bar{F}_p$ , we denote the  $FG$ -set which corresponds to  $E$  by  $T = FG(E)$ . Conversely, if  $T$  is a  $FG$ -set, then  $F_p(T)$  shows the subfield of  $\bar{F}_p$  that is generated by  $T$ .

In [7, Proposition 1.9], it is proved that  $T \subseteq \mathbb{N}$  is a  $FG$ -set if and only if it satisfies in the following conditions:

- (1)  $1 \in T$
- (2) If  $n \in T$  and  $d|n$ , then  $d \in T$ .
- (3) If  $m, n \in T$ , then  $[m, n] \in T$ .

Now we have the following immediate corollary.

**Corollary 1.5.** *A subset  $T$  of  $\mathbb{N}$  is a  $FG$ -set, if and only if there exist disjoint subsets  $A$  and  $B$  of  $\mathbb{P}$  and for each  $p \in A$  there exists a fixed natural number  $n(p)$  such that*

$$T = \{1\} \cup \{p_1^{r_1} \cdots p_m^{r_m} q_1^{s_1} \cdots q_n^{s_n} \mid m, n \in \mathbb{N}, p_i \in A, 0 \leq r_i \leq n(p_i), q_j \in B, s_j \geq 0\}.$$

*Proof.* By the previous comment, it is clear that if  $T$  has the form in the statement of the corollary, then  $T$  is a  $FG$ -set. Conversely, let  $T$  be a  $FG$ -set. Put

$$A = \{p \in \mathbb{P} \mid \exists n \in \mathbb{N}, p^n \in T \text{ but } p^{n+1} \notin T\}$$

and

$$B = \{p \in \mathbb{P} \mid \forall n \in \mathbb{N}, p^n \in T\}$$

also, for each  $p \in A$ , let  $n(p) = \max\{k \in \mathbb{N} \mid p^k \in T\}$ . Then one can easily complete the proof by the comment preceding this corollary.  $\square$

Let us respectively call  $A$  and  $B$  in the previous corollary, the finite and infinite parts of  $T$ ; and denote them by  $T_f$  and  $T_\infty$ , respectively. If  $t \in T$ , then the order of  $t$  in  $T$  which is denoted by  $o_T(t)$ , is the greatest natural number  $n$ , if exists, such that  $t^n \in T$  but  $t^{n+1} \notin T$ , otherwise we define  $o_T(t) = \infty$ , that is  $t^n \in T$  for every natural number  $n$ . Also by the notation of the previous corollary, for each  $q \in T_f$  we have  $o_T(q) = n(q)$  and if  $q \in T_\infty$  we have  $o_T(q) = \infty$ . It is clear that the converse also holds (i.e.,  $q \in T_f$  if and only if  $o_T(q) \in \mathbb{N}$ ). One can easily see that  $t \in T$  has finite order in  $T$  if and only if there exists  $q \in T_f$  such that  $q|t$ . Also note that whenever  $T_1$  and  $T_2$  are  $FG$ -sets, then  $T_1 \subseteq T_2$  if and only if  $o_{T_1}(q) \leq o_{T_2}(q)$  for each prime  $q \in T_1$ . Consequently,  $T_1 \subsetneq T_2$  if and only if there exists a prime  $q \in T_2$  such that either  $q \notin T_1$  or  $o_{T_1}(q) < o_{T_2}(q)$ .

*Remark 1.6.* Let us remind the reader of the correspondence between the  $FG$ -sets and the Steinitz's numbers. If  $S$  is a Steinitz's number of the subfield  $E \subseteq \bar{F}_p$ , then by notation of [14], we have  $T = FG(E) = \{n \in \mathbb{N} \mid n|S\}$ . Conversely, if  $T$  is a  $FG$ -set, then the Steinitz's number of the subfield  $E = F_p(T)$  is  $S = \prod_{p \in \mathbb{P} \cap T} p^{o_T(p)}$ . But we believe the  $FG$ -sets are easier to work with.

Now we need the following definition.

**Definition 1.7.** [7, Definition 1.6]. Let  $T_1 \subset T_2$  be two  $FG$ -sets. Then  $T_1$  is said to be a maximal  $FG$ -subset of  $T_2$  if there is no  $FG$ -set properly between  $T_1$  and  $T_2$ .

By [7, Proposition 1.11] and our new notation we have the following immediate proposition.

**Proposition 1.8.** Let  $T$  be a  $FG$ -set, then  $T'$  is a maximal  $FG$ -subset of  $T$  if and only if there exists a unique prime number  $q \in T_f$  such that  $o_T(q') = o_{T'}(q')$  (hence  $q' \in T'$ ) for each prime  $q' \in T \setminus \{q\}$  and exactly one of the following conditions holds:

- (1)  $o_T(q) = 1$  and  $q \notin T'$ ,
- (2)  $o_{T'}(q) = o_T(q) - 1$ .

In particular,  $T'_\infty = T_\infty$ . Therefore there exists one-one correspondence between  $T_f$  and maximal  $FG$ -subsets of  $T$ . Thus  $T$  has exactly  $|T_f|$ -many maximal  $FG$ -subsets. Moreover by Corollary 1.5,  $T$  has only finitely many maximal  $FG$ -subsets if and only if there exist a natural number  $m$  and a subset  $P$  of prime numbers such that  $m$  has no prime divisor in  $A$  and  $T = \{dn \mid d|m, \text{ and prime divisors of } n \text{ are in } P\} \cup \{1\}$  (note that in this case,  $m = \prod_{p \in T_f} p^{o_T(p)}$ ).

**Corollary 1.9.** Let  $E$  be an absolutely algebraic field of characteristic  $p$  and  $T = FG(E)$ . Then

$$RgMax(E) = \{ F_p(T') \mid T' \text{ is a maximal } FG\text{-subset of } T \}.$$

In particular, there exists a one-one correspondence between  $T_f$  and  $RgMax(E)$ . Therefore  $|RgMax(E)| = |T_f|$ . Hence  $E$  has only finitely many maximal subrings if and only if  $T_f$  is finite, in other words  $E$  has only finitely many maximal subrings if and only if  $E = F_p(T)$  where  $T$  is a  $FG$ -set with only finitely many maximal  $FG$ -subsets.

Hence for an absolutely algebraic field  $E$ , with characteristic  $q$ , we have  $|RgMax(E)| = n \geq 0$  if and only if there exist distinct prime numbers  $p_1, \dots, p_n$ , a subset  $P$  of  $\mathbb{P}$  disjoint from  $\{p_1, \dots, p_n\}$  and natural numbers  $m_i$ ,  $1 \leq i \leq n$ , such that  $E = F_q(T)$  where

$$T = \{1\} \cup \{p_1^{r_1} \cdots p_n^{r_n} q_1^{s_1} \cdots q_k^{s_k} \mid 0 \leq r_i \leq m_i, 1 \leq i \leq n, k \in \mathbb{N}, q_j \in P, s_j \geq 0\}.$$

If we show this kind of  $FG$ -sets by  $T = T(m_1, \dots, m_n)$ , then by Corollary 1.9, we have:

$$RgMax(E) = \{ F_p(T(m_1, \dots, m_{i-1}, m_i - 1, m_{i+1}, \dots, m_n)) \mid 1 \leq i \leq n \},$$

note that in this case  $m_j = 0$  means  $p_j \notin T_f$ .

One can easily see that if  $T$  is a  $FG$ -set,  $T'$  is a maximal  $FG$ -subset of  $T$ ,  $E = F_p(T)$  and  $F = F_p(T')$ , then by the statement of Proposition 1.8, we have  $[E : F] = q$ , see [14, Theorem 2.10].

We remind the reader that whenever  $R \subseteq T$  is an integral extension of rings, then  $|Max(R)| \leq |Max(T)|$ . In particular, if  $T$  is semilocal (resp. local) then  $R$  is semilocal (resp. local). The next example is now in order now.

*Example 1.10.* There exists a field  $E$ , with a unique maximal subring, and containing a subring  $F$ , with infinitely many maximal subrings such that  $E/F$  is an algebraic extension. To see this, let  $P$  be an infinite proper subset of  $\mathbb{P}$  and let  $p$  be a prime number such that  $p \notin P$ ,  $q$  be any prime number, and  $n$  be a fixed natural number. Now, put  $T = \{p^r q_1^{r_1} \cdots q_m^{r_m} \mid 0 \leq r \leq n, r_i \geq 0, m \in \mathbb{N}, q_j \in P\}$ . Clearly,  $T$  is a  $FG$ -set and the field  $E = \bigcup_{n \in T} F_{q^n}$  has a unique maximal subfield, by Proposition 1.8 and Corollary 1.9. For the final part for each  $p' \in P$ , let  $n(p')$  be a fixed natural number and put  $T' = \{p^r q_1^{r_1} \cdots q_m^{r_m} \mid 0 \leq r \leq n, 0 \leq r_j \leq n(q_j), m \in \mathbb{N}, q_j \in P\}$ . It is clear that  $T'$  is a  $FG$ -set with  $T'_f = P \cup \{p\}$  and  $T'_\infty = \emptyset$ . Hence if  $F' = F_q(T')$ , then  $|RgMax(F)| = |P \cup \{p\}| = \aleph_0$ , by Corollary 1.9. It is clear that  $E/F$  is algebraic and therefore we are done.

We recall that each proper ideal  $I$  of a ring  $R$  can be embedded in a maximal one. The natural question which arises from the latter fact is as follow. Whenever  $E$  is a field and  $S$  is a proper subring of  $E$ , can  $S$  be embedded in a maximal subring of  $E$ ? The next example gives a negative answer to this question.

*Example 1.11.* Let  $p_1, p_2$  and  $q$  be prime numbers,  $p_1 \neq p_2$  and  $n > 1$  be any natural number. Put  $T = \{p_1^{m_1} p_2^{m_2} \mid m_1 \geq 0, 0 \leq m_2 \leq n\}$ , it is clear that  $T$  is a  $FG$ -set. Hence  $E = F_q(T)$  is a field. Now, put  $T' = \{p_1^{m_1} p_2^{m_2} \mid m_1 \geq 0, 0 \leq m_2 \leq n-1\}$  and  $E' = F_q(T')$ . By Proposition 1.8,  $T'$  is the unique maximal  $FG$ -subset of  $T$  and therefore  $E'$  is the only maximal subring of  $E$ , by Corollary 1.9. But  $F_{q^{p_2^2}}$  is a subring of  $E$  which clearly is not contained in  $E'$ , by the comments preceding Remark 1.6.

It is well-known and easy to see that no subgroup of  $\mathbb{Q}$  can be embedded in a maximal one, but we have the following interesting fact.

*Remark 1.12.* One can easily see that every proper subring of  $\mathbb{Q}$  can be embedded in a maximal one (note, every subring of  $\mathbb{Q}$  has the form  $\mathbb{Z}_S$ , for some multiplicatively closed subset  $S$  of  $\mathbb{Z}$ ). But  $\mathbb{R}$  does not satisfy this property. To see this, assume that  $E$  is a subfield of  $\mathbb{R}$  such that  $\mathbb{R}/E$  is algebraic. Hence if  $E$  can be embedded in a maximal subring  $R$  of  $\mathbb{R}$ , then  $R$  must be a field (note,  $\mathbb{R}$  is integral over  $R$ ). But  $\mathbb{R}$  has no maximal subring which is a field, by [7, Remark 2.11].

*Remark 1.13. Largest nonsubmaximal subfield:* Assume that  $E$  is an absolutely algebraic field with characteristic  $q$  and  $T = FG(E)$ . Now let  $T'$  be a  $FG$ -subset of  $T$  such that  $T'_\infty = T_\infty$  and  $T'_f = \emptyset$ . Now it is clear that  $L(E) := F_q(T')$  is a nonsubmaximal subfield of  $E$  which contains all nonsubmaximal subfield of  $E$ , by Proposition 1.8, Corollary 1.9 and the comments preceding Remark 1.6. It is clear that  $E$  is not submaximal if and only if  $E = L(E)$ . One can easily see that if  $E/K$  is a finite extension of absolutely algebraic fields with  $T = FG(E)$  and  $T' = FG(K)$ , then  $T_\infty = T'_\infty$  and therefore  $L(E) = L(K)$ .

In the next proposition we characterize absolutely algebraic fields in which every proper subring can be embedded in a maximal one.

**Proposition 1.14.** *Let  $E$  be an absolutely algebraic field with characteristic  $p$  and  $T = FG(E)$ . Then every proper subring of  $E$  can be embedded in a maximal one if and only if  $T_\infty = \emptyset$ , i.e., every element of  $T$  has finite order in  $T$  (or  $L(E) = F_p$ ).*

*Proof.* Assume that  $T_\infty = \emptyset$ . Let  $F$  be a proper subfield (subring) of  $E$  and  $T_1 = FG(F)$ . If  $F$  is a maximal subring of  $E$  we are done. Hence we may assume that  $F$  is not a maximal subring of  $E$  and therefore  $T_1$  is not a maximal  $FG$ -subset of  $T$ , by Corollary 1.9. Thus by Proposition 1.8 and the comments preceding Remark 1.6, we infer that at least one of the following conditions holds:

- (1) there exists a prime  $q \in T_1$  such that  $o_T(q) \geq 2$  and  $o_{T_1}(q) < o_T(q) - 1$ .
- (2) there exist primes  $q_1 \neq q_2$  of  $T$  of order 1 in  $T$  such that  $q_i \notin T_1$ .
- (3) there exist primes  $q_1 \neq q_2$  of  $T$  with  $o_T(q_1) = 1$ ,  $o_T(q_2) = n \geq 2$  and  $q_1 \notin T_1$ ,  $q_2 \in T_1$  but  $o_{T_1}(q_2) = n - 1$ .

Now in case (1), assume that  $T_2$  is a  $FG$ -set generated by the same primes of  $T$  and the order of every prime  $p' \in T_2 \setminus \{q\}$  in  $T_2$  equal to  $o_T(p')$  but  $o_{T_2}(q) = o_T(q) - 1$ . Now it is clear that  $T_2$  is a maximal  $FG$ -subset of  $T$  which contains  $T_1$ , by Proposition 1.8 and the comments preceding Remark 1.6. Hence, if  $E' = F_p(T_2)$  then we infer that  $E'$  is a maximal subfield of  $E$  which contains  $F$ , by Corollary 1.9, Proposition 1.8 and the comments preceding Remark 1.6. Now assume that (2) or (3) holds. Let  $T_2$  be a  $FG$ -set with the same primes of  $T$  except  $q_1$ , i.e.,  $(T_2)_f = T_f \setminus \{q_1\}$  and  $(T_2)_\infty = \emptyset$ , and have the same orders as in  $T$ , i.e., for each  $q \in (T_2)_f$  we have  $o_{T_2}(q) = o_T(q)$ . Then by the same arguments, if  $E' = F_p(T_2)$  then we infer that  $E'$  is a maximal subfield of  $E$  which contains  $F$ . Conversely, assume that every proper subring of  $E$  can be embedded in a maximal one. We show that  $T_\infty = \emptyset$ . To see this, assume that  $q_1$  is a prime of infinite order in  $T$ . Now assume that  $T'$  is a  $FG$ -set which generates by

the same primes in  $T$  and the order of each prime  $q \neq q_1$  in  $T'$  equals to  $o_T(q)$  but  $o_{T'}(q_1) < \infty$  (i.e.,  $T'_f = T_f \cup \{q_1\}$  and  $T'_\infty = T_\infty \setminus \{q_1\}$ ). Now it is clear that for each  $FG$ -subset  $T_1$  such that  $T' \subseteq T_1 \subsetneq T$  we have  $o_T(q) = o_{T'}(q) = o_{T_1}(q)$  for any  $q \neq q_1$  and  $o_{T'}(q_1) \leq o_{T_1}(q_1) < \infty = o_T(q_1)$ , which show that  $T_1$  is not a maximal  $FG$ -subset of  $T$ , by Proposition 1.8. Therefore  $T'$  can not be embedded in a maximal  $FG$ -subset of  $T$ . Consequently, if  $E' = F_p(T')$  then  $E'$  is a proper subfield of  $E$  which is not contained in any maximal subfield of  $E$ , by Corollary 1.9, Proposition 1.8 and the comments preceding Definition 1.7, this is a contradiction and hence we are done.  $\square$

## 2. CHARACTERIZING FIELDS WITH ONLY FINITELY MANY MAXIMAL SUBRINGS

In this section we give some characterizations of fields with only finitely many maximal subrings. We begin with the following fact.

**Corollary 2.1.** [11, Corollary 1.5]. *Let  $K \subseteq E$  be a field extension and  $F$  be the prime subfield of  $E$ . Then the following statements hold:*

- (1) *If  $E$  has zero characteristic, then  $RgMax(E)$  is infinite.*
- (2)  *$|RgMax(E)| \geq \text{tr.deg}(E/K)$ . In particular, if  $E$  is uncountable, then  $|RgMax(E)| \geq |E|$ .*
- (3) *If  $\text{tr.deg}(E/F) \neq 0$ , then  $RgMax(E)$  is infinite.*

*In particular, if  $RgMax(E)$  is finite, then  $E$  is an absolutely algebraic field.*

Now the following theorem, which is one of the main results in this paper, is in order.

**Theorem 2.2.** *Let  $E$  be a field. Then the following conditions are equivalent:*

- (1)  *$E$  has only finitely many maximal subrings.*
- (2)  *$E$  is an absolutely algebraic field and  $[E : L(E)]$  is finite.*
- (3)  *$E$  has a nonsubmaximal subfield  $F$ , such that  $[E : F]$  is finite.*

*In particular, if one of the above conditions holds, then in (3) we have  $F = L(E)$ .*

*Proof.* (1)  $\Rightarrow$  (2) If  $E$  has only finitely many maximal subrings, then by the previous corollary we infer that  $E$  is an absolutely algebraic field with only finitely many maximal subrings. Thus assume that  $E = F_p(T)$ , where  $p$  is the characteristic of  $E$  and  $T = FG(E)$ . Hence by Corollary 1.9, we infer that  $T_f$  is a finite set. Now by [14, Theorem 2.10], we conclude that  $[E : L(E)] = \prod_{q \in T_f} q^{o_T(q)}$  which is finite. Thus (2) holds.

(2)  $\Rightarrow$  (3) is trivial.

(3)  $\Rightarrow$  (1) First note that if  $E$  is not submaximal then (1) holds and we are done. Hence assume that  $E$  is submaximal. Since  $F$  is not submaximal, we infer that  $F$  and thus  $E$  are absolutely algebraic field with characteristic  $q$ , for some prime number  $q$ , by Corollary 2.1. Without loss of generality, we may assume that  $F$  is an infinite field. Now let  $T = FG(F)$ , thus by Corollary 1.9,  $T_f = \emptyset$  and therefore there exists a subset  $P$  of prime numbers such that  $T = \{q_1^{s_1} \cdots q_k^{s_k} \mid s_i \geq 0, q_i \in P, k \in \mathbb{N}\}$  (i.e.,  $T_\infty = P$ ). Now assume that  $T' = FG(E)$ . We claim that  $T'_\infty = P$  and  $T'_f$  is finite. It is obvious that  $P \subseteq T'_\infty$ . Now assume that  $p \in T' \setminus P$  be a prime number. By [14, Theorem 2.10], if  $p^n \in T'$  for some natural number  $n$ , then  $[E : F] \geq p^n$ . Since  $[E : F]$  is finite we infer that  $o_{T'}(p)$  is finite. Thus  $T'_\infty = P$ . Again, since  $[E : F]$  is finite, we infer that  $T'_f$  is finite too (by a similar argument, for each  $p \in T'_f$  we have  $[E : F] \geq p$ ). Thus  $E$  has only finitely many maximal subrings, by Corollary 1.9 and therefore (1) holds. Finally note that the proof of this item shows that  $F = L(E)$  and hence the final assertion holds.  $\square$

**Corollary 2.3.** *Let  $E$  be a field. Then the following conditions are equivalent:*

- (1)  *$E$  has only finitely many maximal subrings.*
- (2)  *$E$  has a nonsubmaximal subfield  $F$  such that  $F \subset E$  is a FIP-extension.*
- (3)  *$E$  has a nonsubmaximal subfield  $F$  such that  $F \subset E$  is a finite simple extension (i.e.,  $E = F[\alpha]$ , for some  $\alpha \in E$ ).*

*Proof.* If (1) holds, then by the previous theorem let  $F = L(E)$  and  $T = FG(E)$ . Thus by Corollary 1.9,  $T_f$  is finite. Now note that if  $K$  is a subring (i.e., a subfield) of  $E$  such that  $F \subseteq K \subseteq E$ , then since  $[E : F]$  is finite, similar to the proof of (2)  $\Rightarrow$  (3) in the previous theorem we infer that  $T'_\infty = T_\infty$  and  $T'_f \subseteq T_f$ , where  $T' = FG(K)$  (also note that for each  $q \in T'_f$  we have  $o_{T'}(q) \leq o_T(q)$ ). This immediately implies that there are only finitely many subrings (or subfield) between  $F$  and  $K$ . Thus (2) holds. Conversely, if (2) holds, then there exists a subfield  $F$  of  $E$  which is not submaximal and  $F \subseteq E$  is a FIP-extension. Then clearly  $E$  is algebraic over  $F$  and  $[E : F]$  is finite, hence we are done by the

previous theorem. Conditions (2) and (3) are equivalent by Steinitz's Primitive Element Theorem, see [15, Theorem 7.9.3].  $\square$

**Corollary 2.4.** (1) *Let  $\mathcal{E}$  be the set of all fields, up to isomorphism, with only finitely many maximal subrings. Then  $|\mathcal{E}| = 2^{\aleph_0}$ .*

(2) *Let  $\mathcal{E}'$  be the set of all fields, up to isomorphism, which are submaximal but have only finitely many maximal subrings. Then  $|\mathcal{E}'| = 2^{\aleph_0}$ .*

*Proof.* Let  $\mathcal{F}$  be the set of all fields, up to isomorphism, without maximal subrings. Then by [7, Corollary 1.5], we have  $|\mathcal{F}| = 2^{\aleph_0}$ . Now note that  $\mathcal{F} \subseteq \mathcal{E}$  and every element of  $\mathcal{E}$  is an absolutely algebraic field. Thus we conclude that  $|\mathcal{E}| = 2^{\aleph_0}$ . This proves (1). For (2) note that  $E \in \mathcal{E}'$  if and only if  $E$  is an absolutely algebraic field with  $T_\infty \subseteq \mathbb{P}$  and  $T_f$  is a finite nonempty subset of  $\mathbb{P}$  which is disjoint from  $T_\infty$ , where  $T = FG(E)$ . This immediately implies that  $|\mathcal{E}'| = 2^{\aleph_0}$ .  $\square$

Before presenting the next characterization we need some observations. First we recall the following fact from [11].

**Corollary 2.5.** [11, Corollary 1.12]. *Let  $E$  be a field which either is not algebraic over its prime subfield or has zero characteristic. Then there exists an infinite chain  $\cdots \subset R_2 \subset R_1 \subset R_0 = E$  where each  $R_i$  is a non-field  $G$ -domain maximal subring of  $R_{i-1}$ ,  $i \geq 1$ .*

For the next result we need some notation. Let  $A = \{p_i\}_{i \in I}$  (where if  $I$  is not empty, then we assume that either  $I = \{1, \dots, n\}$  or  $I = \mathbb{N}$ ) and  $B$  be two disjoint subset of  $\mathbb{P}$  and for each  $i \in I$  let  $m_i \geq 0$  be a fixed integer. Now by the comments preceding Corollary 1.5, one can easily see that the set

$$T = \{p_{i_1}^{r_{i_1}} \cdots p_{i_k}^{r_{i_k}} q_1^{s_1} \cdots q_n^{s_n} \mid k, n \in \mathbb{N}, i_j \in I, 0 \leq r_{i_j} \leq m_{i_j}, q_a \in B, s_a \geq 0\}$$

is a  $FG$ -set and we have  $T_\infty = B$  and  $T_f = \{p_i \in A \mid m_i > 0\}$ . Let us denote such  $FG$ -sets by  $T = T(m_1, \dots, m_n)$  when  $I = \{1, \dots, n\}$ , and otherwise by  $T = T(m_1, m_2, \dots)$ . It is clear that by Proposition 1.8, maximal  $FG$ -subsets of  $T$  have the form  $T' = T(m_1, m_2, \dots, m_{i-1}, m_i - 1, m_{i+1}, \dots)$ , where  $m_i > 0$ .

The following which is another main result in this section, resembles the well-known fact that artinian rings have only finitely many maximal ideals and all composition series in these rings have the same length.

**Theorem 2.6.** *Let  $E$  be a field, then the following conditions are equivalent:*

- (1)  *$E$  has only finitely many maximal subrings.*
- (2) *Every descending chain*

$$\cdots \subset R_2 \subset R_1 \subset R_0 = E$$

*where each  $R_i$  is a maximal subring of  $R_{i-1}$ ,  $i \geq 1$ , is finite.*

*Moreover, if one of the above equivalent conditions holds, then all chains in (2) have the same length,  $m$  say, and  $R_m = L(E)$ . Furthermore,  $E$  has only finitely many chains of this form and  $m = \sum_{p \in T_f} o_T(p)$ , where  $T = FG(E)$ .*

*Proof.* Assume (1) holds, thus by Corollary 2.1,  $E$  is an absolutely algebraic field. Let  $T = FG(E)$  and  $q$  be the characteristic of  $E$ . Hence by the above notation and by Corollary 1.9, we infer that  $T = T(m_1, \dots, m_n)$  for some  $n$  and  $m_i \geq 0$ . Now since  $R_1$  is a maximal subring of  $R_0 = E$ , by Proposition 1.8 and Corollary 1.9, we conclude that  $R_1 = F_q(T_1)$ , where  $T_1 = T(m_1, \dots, m_i - 1, \dots, m_n)$ , for some  $i$ ,  $1 \leq i \leq n$ , and  $m_i > 0$  (note,  $(T_1)_\infty = T_\infty$ ). Therefore, we deduce that this chain will stop after  $m$  steps, where  $m = m_1 + m_2 + \cdots + m_n$  and  $R_m = T(0, \dots, 0) = L(E)$ , by the previous notation. Also note that for each  $i \geq 0$ ,  $R_i$  has only finitely many maximal subrings, i.e., we have finitely many choices for  $R_{i+1}$ . Thus  $E$  has only finitely many chains of this form. This proves (2) and the final assertions of the theorem. Conversely, assume that (2) holds. It is clear that  $E$  is algebraic over  $F_q$  for some prime number  $q$ , by the previous corollary. Now suppose that  $T = FG(E)$ . To prove (1), by Corollary 1.9, it suffices to show that  $T_f$  is finite. By the way of contradiction, assume that  $T_f = \{p_i\}_{i=1}^\infty$  and  $o_T(p_i) = m_i > 0$ , for each  $i \in \mathbb{N}$ . Now by the above notation, Proposition 1.8 and Corollary 1.9, if we put  $R_0 = F_q(T(m_1, m_2, \dots)) = E$  and  $R_i = F_q(T(m_1 - 1, m_2 - 1, \dots, m_i - 1, m_{i+1}, \dots))$  we have an infinite descending saturated chain of maximal subrings  $\cdots \subset R_2 \subset R_1 \subset R_0 = E$ , which is a contradiction. Thus  $T_f$  is finite and we are done.  $\square$

*Example 2.7.* There exists a field  $E$  with a unique maximal subring such that  $E$  has an infinite saturated chain  $\cdots R_{-2} \subset R_{-1} \subset R_0 \subset R_1 \subset R_2 \subset \cdots$ , of subrings where each  $R_i$  is a maximal subring of  $R_{i+1}$ , for  $i \in \mathbb{Z}$ . To see this, let  $p, p_1, p_2, \dots$  be distinct prime numbers, and  $q$  be any prime number. Now put  $T = \{p^r p_1^{r_1} \cdots p_m^{r_m} \mid 0 \leq r \leq k, r_i \geq 0, m \in \mathbb{N}\}$ , where  $k$  is a fixed natural number, it is clear that  $T$  is a  $FG$ -set. Hence  $E = F_q(T)$  is a field with unique maximal subring, by Proposition 1.8 and Corollary 1.9. Now for  $n \geq 0$  put

$$T_{-n} = \{p_{2n+1}^{r_{2n+1}} p_{2n+3}^{r_{2n+3}} \cdots p_{2m+1}^{r_{2m+1}} \mid 0 \leq r_i \leq 1, n \leq m \in \mathbb{N}\}$$

and for  $n \geq 1$  put

$$T_n = \{p_2^{r_2} \cdots p_{2n}^{r_{2n}} p_1^{r_1} p_3^{r_3} \cdots p_{2m+1}^{r_{2m+1}} \mid 0 \leq r_i \leq 1, m \geq 0\}.$$

It is clear that for each integer  $n$ ,  $T_n$  is a  $FG$ -set and  $T_n$  is a maximal  $FG$ -subset of  $T_{n+1}$ , by Proposition 1.8. Hence if  $R_n = F_q(T_n)$ , then we infer that each  $R_n$  is a subfield of  $E$  which is a maximal subring of  $R_{n+1}$ , for each  $n \in \mathbb{Z}$ , by Corollary 1.9.

We recall that an extension  $S \subseteq R$  of rings is called a  $FCP$ -extension if every chain of subrings between  $S$  and  $R$  is finite, see [20]. It is clear that in this case each subring between  $S$  and  $R$  is affine over  $S$  (such extension is called strongly affine). Now we have the following corollary whose proof is simple and is left to the reader.

**Corollary 2.8.** *Let  $E$  be a field. Then the following conditions are equivalent:*

- (1)  *$E$  has only finitely many maximal subrings.*
- (2)  *$E$  has a nonsubmaximal subfield  $F$  such that  $F \subseteq E$  is a  $FCP$ -extension.*
- (3)  *$E$  has a nonsubmaximal subfield  $F$  such that  $E/F$  is algebraic and every chain  $F = R_0 \subset R_1 \subset \cdots \subset E$  where each  $R_i$  is a maximal subring of  $R_{i+1}$ , is finite.*
- (4)  *$E$  has a nonsubmaximal subfield  $F$  such that there exists a finite chain  $F = R_0 \subset R_1 \subset \cdots \subset R_n = E$  where each  $R_i$  is a maximal subring of  $R_{i+1}$ .*

Moreover, all of the above conditions are equivalent if in the conditions (2) – (4) we replace  $F$  by  $L(E)$ . Furthermore, in fact if one of the above conditions holds then  $F = L(E)$  and all chains have the same length.

*Example 2.9.* The algebraic condition in (3) of the above corollary is needed. To see this, let  $F$  be an absolutely algebraic field which is algebraically closed and  $E = F(x)$ . Then one can easily see that  $F$  is the largest nonsubmaximal subfield of  $E$ . Also, there exists no chain  $F = R_0 \subset R_1 \subset \cdots \subset E$ , where  $R_i$  is a maximal subring of  $R_{i+1}$ ; for  $R_1$  is algebraic over  $F$  (note, for each  $x \in R_1$  either  $x^2 \in F$  or  $x \in F[x^2]$ ) and therefore  $R_1 = F$ . But  $E$  has infinitely many maximal subrings by Corollary 2.1.

The next remark gives a natural characterization of fields with only finitely many maximal subrings.

*Remark 2.10.* Let  $R$  be a ring and  $X = RgMax(R)$ . Then we have a topology on  $X$ , by putting  $\mathbb{X}(S) = \{T \in X \mid S \subseteq T\}$  as a subbase for closed subsets for  $X$ , where  $S$  ranges over all subrings of  $R$ , which is called  $K$ -space in [4]. In [4], it is shown that if  $E$  is a field then  $X = RgMax(E)$  is compact if and only if  $E$  has only finitely many maximal subrings.

The following corollary which is an application of the Theorems 2.2 and 2.6 will be used for the next section.

**Corollary 2.11.** *Let  $E \subseteq K$  be a finite extension of fields. Then  $E$  has only finitely many maximal subrings if and only if  $K$  has only finitely many maximal subrings.*

*Proof.* First assume that  $K$  has only finitely many maximal subrings. Since  $K/E$  is a finite extension of fields, we infer that there exists a finite chain  $E = T_m \subset T_{m-1} \subset \cdots \subset T_2 \subset T_1 \subset T_0 = K$ , where each  $T_i$  is a maximal subring of  $T_{i-1}$ . Thus we conclude that every chain  $\cdots \subset R_2 \subset R_1 \subset R_0 = E$ , where each  $R_i$  is a maximal subring of  $R_{i-1}$ , can be enlarged to a saturated descending chain of maximal subrings which begins from  $K$ . Thus by Theorem 2.6, we deduce that the chain is stationary and therefore by Theorem 2.6,  $E$  has only finitely many maximal subrings. Conversely, assume that  $E$  has only finitely many maximal subrings. By Theorem 2.2, we infer that  $[E : L(E)]$  is finite. Thus by our assumption, we infer that  $[K : L(E)]$  is finite and therefore by (3) of Theorem 2.2, we conclude that  $K$  has only finitely many maximal subrings.  $\square$



## 3. AFFINE RINGS

In this section we study certain affine rings with only finitely many maximal subrings. Before presenting the next main result in this article, let us recall the important Zariski's Lemma (which play a key role in the proof of Hilbert's Nullstellensatz Theorem, see [33]) which say an affine integral domain  $R = F[\alpha_1, \dots, \alpha_n]$  over a field  $F$  is a field if and only if each  $\alpha_i$  is algebraic over  $F$ . One can easily see that in fact this lemma is also valid if instead of assuming that  $R$  is a field we just assume that  $R$  is semilocal (i.e., an affine integral domain  $R = F[\alpha_1, \dots, \alpha_n]$  over a field  $F$  is semilocal if and only if each  $\alpha_i$  is algebraic over  $F$  and therefore  $R$  is a field too). More generally, in the light of [24, Theorem 22], one also can prove that if  $T$  is an integral domain and  $T = R[\alpha_1, \dots, \alpha_n]$ , then  $T$  is a  $G$ -domain (field) if and only if  $R$  is a  $G$ -domain and each  $\alpha_i$  is algebraic over  $R$ . The following result is a similar result for maximal subrings.

**Theorem 3.1.** *Let  $F \subseteq E$  be an extension of fields and  $\alpha_1, \dots, \alpha_n \in E$ . Then*

- (1)  *$K = F(\alpha_1, \dots, \alpha_n)$  has only finitely many maximal subrings if and only if  $F$  has only finitely many maximal subrings and  $K/F$  is finite.*
- (2)  *$R = F[\alpha_1, \dots, \alpha_n]$  has only finitely many maximal subrings if and only if  $F$  has only finitely many maximal subrings and each  $\alpha_i$  is algebraic over  $F$  (i.e.,  $R/F$  is a finite extension of fields).*

*Proof.* (1) If  $K/F$  is finite then we are done by Corollary 2.11. Hence assume that  $K$  has only finitely many maximal subrings. Thus by (3) of Corollary 2.1,  $K$  is algebraic over  $F$ . Therefore  $K$  is finite over  $F$  and hence by Corollary 2.11,  $F$  has only finitely many maximal subrings.

(2) If each  $\alpha_i$  is algebraic over  $F$ , then we are done by (1). Hence assume that  $R$  has only finitely many maximal subrings. First note that if  $M$  is a maximal ideal of  $R$ , then  $F \subseteq K_M := R/M = F[\bar{\alpha}_1, \dots, \bar{\alpha}_n]$  is a field extension, where  $\bar{\alpha}_i = \alpha_i + M$ . Now since  $R$  has only finitely many maximal subrings, we infer that  $K_M$  has only finitely many maximal subrings. Therefore by the previous part we infer that  $F$  has only finitely many maximal subrings and  $K_M/F$  is finite. For the final assertion, by the previous part we may assume that  $R$  is not a field. Hence  $\text{tr.deg}(R/F) = m > 0$ . Thus by Noether's Normalization Theorem, there exist  $x_1, \dots, x_m \in R$  which are algebraically independent over  $F$  and  $R$  is integral (and therefore finite as module) over  $S = F[x_1, \dots, x_m]$ . Now since  $\text{Max}(S)$  is infinite we infer that  $\text{Max}(R)$  is infinite too. Hence assume that  $M_1, M_2, \dots$  is a sequence of distinct maximal ideals of  $R$  and  $K_i = R/M_i$ . By the first part of the proof of this item, each  $K_i$  is a finite field extension of  $F$ . Thus if  $K_i \neq F$  for infinitely many  $i \in I \subseteq \mathbb{N}$ , we infer that for each  $i \in I$ ,  $K_i$  has a maximal subrings; i.e.,  $R$  has a maximal subrings  $S_i$  which contains  $M_i$ , for each  $i \in I$ . Now note that since  $M_i + M_j = R$  for  $i \neq j$  in  $I$ , we deduce that  $S_i \neq S_j$ . Thus  $R$  has infinitely many maximal subrings which is a contradiction. Hence we conclude that there exists  $k$ , such that for each  $r \geq k$ , we have  $K_r \cong F$ . Hence for each distinct  $r, s \geq k$ , we conclude that  $R/(M_r \cap M_s) \cong F \times F$ , which by [2, Theorem 2.2], immediately implies that  $R$  has a maximal subrings which contains  $M_r \cap M_s$ . Now put  $I_j = M_{2j} \cap M_{2j+1}$ , for  $j \geq k$ . Thus we deduce that  $R$  has a maximal subring  $T_j$  which contains  $I_j$ , for each  $j \geq k$ . Now since  $I_j + I_{j'} = R$ , for distinct  $j, j' \geq k$ , we conclude that  $T_j \neq T_{j'}$ , i.e.,  $R$  has infinitely many maximal subrings which is a contradiction. Hence  $\text{tr.deg}(R/F) = 0$ , i.e., each  $\alpha_i$  is algebraic over  $F$  and we are done.  $\square$

By the previous theorem and Corollary 2.1, we have the following corollary.

**Corollary 3.2.** *Let  $F$  be an algebraically closed field and  $R$  be an affine integral domain over  $F$ . Then  $R$  has only finitely many maximal subrings if and only if  $R = F = \bar{F}_p$  for some prime number  $p$ . In particular in this case  $R$  has no maximal subrings.*

We remind the reader that by [11, Corollary 3.5], for a field  $K$ , the ring  $K \times K$  has only finitely many maximal subrings if and only if  $K$  is finite.

**Proposition 3.3.** *Let  $F$  be a field and  $R = F[\alpha_1, \dots, \alpha_n]$  be a reduced  $F$ -algebra. If  $R$  has only finitely many maximal subrings, then the following statements hold:*

- (1)  *$F$  has only finitely many maximal subrings.*
- (2)  *$R \cong K_1 \times \dots \times K_m$ , where each  $K_i$  is a finite field extension of  $F$  (therefore each  $K_i$  has only finitely many maximal subrings). Moreover, if  $K_i$  is infinite, then  $K_i \not\cong K_j$  for each  $j \neq i$ .*

*Proof.* First note that for each maximal ideal  $M$  of  $R$ , the field  $R/M$  (which has only finitely many maximal subrings) is finite over  $F$ , therefore we infer that  $F$  has only finitely many maximal subrings, by Corollary 2.11; and similarly to the proof of (2) of Theorem 3.1, we infer that  $R$  is a semilocal ring.

Now note that since  $R$  is a Hilbert ring we infer that  $J(R) = 0$  and therefore  $R \cong K_1 \times \cdots \times K_m$ , where each  $K_i$  is a finite field extension of  $F$ . The final part of (2) is evident by the above comment.  $\square$

**Corollary 3.4.** *Let  $F$  be a field and  $V$  be an affine variety in  $A^n(F)$ . If the coordinate ring  $F[V]$  of  $V$  has only finitely many maximal subrings, then  $V$  is finite. Moreover in this case either  $F[V]$  is finite or  $F[V] = F$  (and therefore  $|V| = 1$ ).*

*Proof.* Since  $F[V]$  is a reduced finitely generated  $F$ -algebra, by the previous proposition, we infer that  $F[V]$  is semilocal and  $F$  has only finitely many maximal subrings. But for each  $P \in V$ ,  $M_P/I(V)$  is a maximal ideal of  $F[V]$  hence we infer that  $V$  is finite. Hence if  $V = \{P_1, \dots, P_n\}$ , then  $I(V) = M_{P_1} \cap \cdots \cap M_{P_n}$  and thus  $F[V] \cong \prod_{i=1}^n F$ . Therefore, if  $F$  is finite we conclude that  $F[V]$  is finite too, and if  $F$  is infinite, then by the comment preceding Corollary 3.3, we infer that  $n = 1$ , i.e.,  $V$  is a singleton.  $\square$

As an application of the previous results, we prove the following interesting fact which is in [18, Proposition V.1].

**Corollary 3.5.** *Let  $R$  be a ring with nonzero characteristic which has only finitely many subrings, then  $R$  is finite.*

*Proof.* Let  $\text{Char}(R) = n$ , since  $R$  has finitely many subring, we infer that  $R = \mathbb{Z}_n[b_1, \dots, b_n]$ , for some  $b_i \in R$ . Thus  $R$  is a Hilbert ring. Since  $R$  has finitely many subrings, for each maximal ideals  $M$  of  $R$ , we infer that  $R/M$  is a finite field. Now we have two cases either  $R$  is a semilocal ring or not. If  $R$  is semilocal ring, since  $R$  is a Hilbert ring, then we immediately conclude that  $R$  is a zero dimensional ring, which by [11, Proposition 2.1], we infer that  $R$  is integral over  $\mathbb{Z}_n$ , since  $R$  has finitely many maximal subrings. Thus  $R$  is finite in this case. Hence assume that  $R$  has infinitely many maximal ideals. Let  $M_1, M_2, \dots$  be a sequence of distinct maximal ideals of  $R$ . Thus for each  $i$ , the field  $K_i = R/M_i$  is a finite extension of  $F_{p_i}$  for some prime number  $p_i$ , where  $p_i | n$ . Now similar, to the proof of (2) of Theorem 3.1, we infer that there exists  $n$  such that for each  $i \geq n$  we have  $K_i = F_{p_i}$ . Since for each  $i$ ,  $p_i | n$ , we conclude that there exists a prime number  $p$  (where  $p | n$ ) and a sequence  $n < r_1 < r_2 < \cdots$  such that  $K_{r_i} = F_p$  for each  $i$ . Therefore for each  $i \neq j$  we have  $R/(M_{r_i} \cap M_{r_j}) \cong F_p \times F_p$ . Again similar to the proof of (2) of Theorem 3.1, we infer that  $R$  has infinitely many maximal subrings which is absurd.  $\square$

By the above corollary and [11, Proposition 2.1], one can easily deduce that if  $R$  is a zero-dimensional ring with only finitely many subrings, then  $R$  is a finite ring.

**Lemma 3.6.** *Let  $K$  be a field and  $x$  be an indeterminate over  $K$ . Then any subring  $R$ , where  $K \subsetneq R \subsetneq K[x]$  is affine over  $K$  (thus  $R$  is noetherian) and  $K[x]$  is integral over  $R$ . Moreover,  $R$  has a maximal subring  $T \neq K$ . Consequently, there exists an infinite chain  $K \subsetneq \cdots \subset R_1 \subset R_0 = K[x]$ , where each  $R_i$  is a maximal subring of  $R_{i-1}$  and  $K[x]$  is integral over each  $R_i$ , for  $i \geq 1$ .*

*Proof.* First we prove that if  $K \subseteq R \subseteq K[x]$ , then  $R$  is affine over  $K$ . It is true when  $R = K$ , hence assume that  $R \neq K$ . Thus there exists a non constant polynomial  $f(x) \in K[x]$  such that  $r_0 = f(x) \in R$ . Now put  $F(t) = f(t) - r_0$ . Since  $K \subseteq R$  and  $r_0 \in R$  we infer that  $F(t) \in R[t]$ . Now  $F(x) = 0$  and  $K \subseteq R$  immediately imply that  $x$  is integral over  $R$ . Again, since  $K \subseteq R$ , we have  $R[x] = K[x]$ , i.e.,  $K[x]$  is a finitely generated  $R$ -module. Thus  $K[x]$  is integral over  $R$  and by Artin-Tate Theorem we infer that  $R$  is an affine domain over  $K$  which by [2, Corollary 2.7], immediately implies that  $R$  has a maximal subring  $T$  which contains  $K$ . Also note that since  $R$  is algebraic over  $T$ , we infer that  $K \neq T$ . This and [2, Corollary 2.7], immediately imply the final assertion of the lemma.  $\square$

**Corollary 3.7.** *Let  $R$  be a ring and  $x$  be an indeterminate over  $R$ . Then there exists an infinite chain  $\cdots \subset R_1 \subset R_0 = R[x]$ , where each  $R_i$  is a maximal subring of  $R_{i-1}$  and  $R[x]$  is integral over each  $R_i$ , for  $i \geq 1$ .*

We remind the reader that by [11, Corollary 1.9], for each ring  $R$ , either  $R$  has infinitely many maximal subrings or  $R$  is a Hilbert ring. Now the following proposition is now in order.

**Proposition 3.8.** *Let  $K$  be an algebraically closed field and  $R$  be an  $K$ -algebra. Then either  $R$  has infinitely many maximal subrings or  $K = \bar{F}_p$ , for some prime number  $p$ , in which case  $R$  is a zero dimensional ring with unique prime ideal  $M$  such that  $R/M \cong K$  and  $R$  is integral over  $F_p$ . In particular, if  $R$  is an integral domain then  $R = K$ .*

*Proof.* If  $K$  is not an absolutely algebraic field, then for each maximal ideal  $M$  of  $R$ , since  $R/M$  contains a copy of  $K$ , we infer that  $R/M$  is not absolutely algebraic field and therefore  $R/M$  has infinitely many maximal subbrings, by Corollary 2.1. Thus  $R$  has infinitely many maximal subbrings and we are done. Hence assume that  $K$  is an absolutely algebraic field and hence there exists a prime ideal  $p$  such that  $K = \bar{F}_p$ . Now assume that  $R$  has finitely many maximal subbrings. Hence we infer that for each maximal ideal  $M$  of  $R$ , the field  $R/M$  has only finitely many maximal subbrings, thus by Corollary 2.1,  $R/M$  is an absolutely algebraic field which also contains a copy of  $K$ . Therefore we conclude that  $R/M \cong K$ . Now if  $R$  has two distinct maximal ideals, say  $M$  and  $N$ , then we infer that  $R/(M \cap N) \cong K \times K$ , which by the comment preceding Corollary 3.3, immediately implies that  $R$  has infinitely many maximal subbrings which is absurd. Thus we infer that  $R$  is a local ring with unique maximal ideal  $M$  and  $R/M \cong K$ . Now, by the above comment  $R$  is a Hilbert ring. Thus we conclude that  $R$  is a zero dimensional ring, and therefore by [11, Proposition 2.1],  $R$  is integral over  $F_p$ . The final part is evident.  $\square$

**Theorem 3.9.** *Let  $R \subseteq T$  be an extension of rings and  $T = R[\alpha_1, \dots, \alpha_n]$ . Assume that  $T$  has only finitely many maximal subbrings. Then the following statements hold:*

- (1)  *$R$  is zero-dimensional if and only if  $T$  is zero-dimensional.*
- (2)  *$R$  is semilocal (resp. artinian) if and only if  $T$  is semilocal (resp. artinian).*

*Moreover, in any case  $T$  is a finitely generated  $R$ -module and for each prime ideal  $P$  of  $R$ , the ring  $R/P$  has only finitely many maximal subbrings. Furthermore, in case (2),  $R/N(R)$  has only finitely many maximal subbrings up to isomorphism.*

*Proof.* (1) First note that if  $T$  is a zero-dimensional ring with finitely many maximal subbrings, then by [11, Proposition 2.1 or Corollary 2.2],  $T$  has a nonzero characteristic,  $m$  say, and  $T$  is integral over  $\mathbb{Z}_m$ . Thus  $T$  is integral over  $R$ . Therefore  $R$  is zero-dimensional and clearly,  $T$  is a finitely generated  $R$ -module, since  $T$  is affine and integral over  $R$ . Conversely, assume that  $R$  is zero-dimensional. Let  $Q$  be a prime ideal of  $T$ , thus  $P = Q \cap R$  is a maximal ideal of  $R$ . Since  $T$  is affine over  $R$ , we infer that the integral domain  $T/Q$  is affine over the field  $R/P$ , which by (2) of Theorem 3.1, immediately implies that  $T/Q$  is a field (and  $R/P$  has only finitely many maximal subbrings), for  $T/Q$  has only finitely many maximal subbrings. Thus  $T$  is zero-dimensional and note that by the first part we conclude that  $T$  is finitely generated as an  $R$ -module.

(2) If  $T$  is semilocal (resp. artinian), then by the comment preceding Proposition 3.8, we infer that  $T$  is Hilbert. Therefore  $T$  is zero-dimensional, which by the previous case immediately implies that  $T$  is a finitely generated  $R$ -module and therefore  $R$  is semilocal (resp.  $R$  is artinian by Eakin-Nagata Theorem). Conversely, assume that  $R$  is semilocal (resp. artinian), we show that  $T$  is semilocal (resp. artinian) too. First note that if  $M$  is a maximal ideal of  $T$ , then  $T/M$  is an absolutely algebraic field by Corollary 2.1, since  $T$  has only finitely many maximal subbrings. This immediately implies that every subring of  $T/M$  is a field and therefore  $(R + M)/M$  is a subfield of  $T/M$ , i.e.,  $R \cap M$  is a maximal ideal of  $R$ . Also note that the field  $T/M$  is affine over the field  $R/(R \cap M)$ , therefore  $T/M$  is a finite field extension of  $R/(R \cap M)$ . Now, similarly to the proof of Theorem 3.1, if  $T$  has infinitely many maximal ideals, since  $R$  is semilocal, then we conclude that there exist distinct maximal ideals  $M_1, \dots, M_k, \dots$ , such that  $R \cap M_i = N$ , for each  $i$ . Therefore  $T/M_i$  is a finite field extension of  $F = R/N$ . Again similar to the proof of Theorem 3.1, we infer that  $T$  has infinitely many maximal subbrings which is a contradiction. Hence  $T$  is semilocal. Note that if  $R$  is artinian then  $R$  is semilocal and by the previous proof  $T$  is semilocal. Now by the first part of the proof of this item we deduce that  $T$  is zero-dimensional which is finitely generated as an  $R$ -module. This immediately implies that  $T$  is an artinian  $R$ -module and hence  $T$  is an artinian ring. Finally, note that in any case in the above proofs,  $T$  is a zero-dimensional ring which is a finitely generated  $R$ -module. Thus  $R$  is zero-dimensional too. Consequently, for each prime ideal  $P$  of  $R$ , we infer that there exists a prime ideal  $Q$  of  $T$  such that  $R \cap Q = P$ , which by the proof of (1), we deduce that  $R/P$  has only finitely many maximal subbrings. In case (2), note that  $T/N(T)$  is a semilocal reduced ring with only finitely many maximal subbrings and  $R/N(R)$  is a subring of  $T/N(T)$ , which by [11, Corollary 3.21], we conclude that  $R/N(R)$  has only finitely many maximal subbrings up to isomorphism (note,  $T/N(T)$  is a finitely generated as  $R/N(R)$ -module).  $\square$

In the following example we show that in the condition (2) of the previous theorem, the finite condition on the set of maximal subbrings of  $R$  or  $T$  can not be shared between  $R$  and  $T$ .

*Example 3.10.* Let  $K$  be an infinite field without maximal subbrings which is not algebraically closed.

- (1) Assume that  $R = K \times K$ , then by [11, Corollary 3.5],  $R$  has infinitely many maximal subbrings. Now let  $\alpha$  and  $\beta$  be elements of algebraic closure of  $K$  with different degrees over  $K$ . Hence

$K[\alpha] \not\cong K[\beta]$  and therefore by [11, Corollary 3.7], the ring  $T = K[\alpha] \times K[\beta]$  has only finitely many maximal subrings (note,  $K[\alpha]$  and  $K[\beta]$  have only finitely many maximal subrings by Corollary 2.11). It is clear that  $T = R[(\alpha, \beta)]$ .

- (2) Assume that  $R = K$  and  $T = K \times K$ . Clearly  $T = R[(1, 0)]$ ; as we see in (1),  $T$  has infinitely many maximal subrings but  $R$  has no maximal subrings.

Let  $K$  be a field, then in [21, Lemma 1.2] it is shown that the minimal ring extensions of  $K$ , up to  $K$ -algebra, isomorphism are as follow:

- (1) a finite minimal field extension  $E$ .
- (2)  $K \times K$ .
- (3)  $K[x]/(x^2)$ .

Conversely, in [11, Theorem 3.4], it is proved that  $R$  is a maximal subring of  $K \times K$  if and only if  $R$  satisfies in exactly one of the following conditions:

- (1)  $R = S \times K$  or  $R = K \times S$ , for some  $S \in \text{RgMax}(K)$ .
- (2)  $R = \{(\sigma_1(x), \sigma_2(x)) \mid x \in K\}$ , where  $\sigma_i \in \text{Aut}(K)$  for  $i = 1, 2$ .

In the next theorem we determine exactly maximal subrings of  $K[x]/(x^2)$ . We recall that if  $\sigma \in \text{Aut}(K)$ , then the additive map  $\delta : K \rightarrow K$  is called a  $\sigma$ -derivation of  $K$  if for each  $x, y \in K$ , we have  $\delta(xy) = \sigma(x)\delta(y) + \sigma(y)\delta(x)$ . One can easily see that for each nonzero element  $x$  of  $K$  we have  $\delta(x^{-1}) = -\delta(x)\sigma(x)^{-2}$ . In [21], it is shown that if  $R$  is a maximal subring of  $T$ , then  $(R : T) := \{x \in T \mid Tx \subseteq R\}$  is a prime ideal of  $R$ . Moreover,  $T$  is integral over  $R$  if and only if  $(R : T) \in \text{Max}(R)$ ; and otherwise (i.e.,  $R$  is integrally closed in  $T$ ) we have  $(R : T) \in \text{Spec}(T)$ . Now the following is in order.

**Theorem 3.11.** *Let  $K$  be a field and  $T = K[x]/(x^2)$  ( $= K[\alpha]$ , where  $\alpha = x + (x^2)$ ). Then  $R$  is a maximal subring of  $T$  if and only if  $R$  satisfies in exactly one of the following conditions:*

- (1)  $R = S + K\alpha$ , for  $S \in \text{RgMax}(K)$ .
- (2)  $R = \{\sigma(x) + \delta(x)\alpha \mid x \in K\}$ , where  $\sigma \in \text{Aut}(K)$  and  $\delta$  is a  $\sigma$ -derivation of  $K$ .

*Proof.* First assume that  $R$  satisfies one of the above conditions. We show that  $R$  is a maximal subring of  $T$ . It is clear that if  $R$  satisfies in condition (1) then  $R$  is a maximal subring of  $T$ . Hence assume that  $R$  satisfies in condition (2). One can easily see that  $R$  is a subring of  $T$ . Now we prove that  $R$  is a field. For proof note that for each nonzero element  $x$  of  $K$  we have  $(\sigma(x) + \delta(x)\alpha)^{-1} = \sigma(x^{-1}) - \delta(x^{-1})\alpha$  which is an element of  $R$ . This immediately implies that  $R$  is a field. Now note that the function  $f : K \rightarrow R$  where  $f(x) = \sigma(x) + \delta(x)\alpha$  is a ring homomorphism which clearly is one-one and onto. Thus  $f$  is an isomorphism and therefore  $R \cong K$ . Now since  $\alpha^2 = 0$  and  $R$  is a field we infer that  $\alpha \notin R$  and therefore  $R[\alpha] = R \oplus R\alpha$ . Since  $R\alpha = \{\sigma(x)\alpha \mid x \in K\} = K\alpha \subseteq R[\alpha]$ , we immediately conclude that for each  $x \in K$  we have  $\sigma(x) \in R[\alpha]$ , i.e.,  $K \subseteq R[\alpha]$  and therefore  $T = K \oplus K\alpha \subseteq R[\alpha]$ . Hence  $T = R[\alpha] = R \oplus R\alpha$ . Thus  $T$  is a two dimensional vector space over  $R$ , which immediately implies that  $R$  is a maximal subring of  $T$ .

Conversely, assume that  $R$  is a maximal subring of  $T$ . Since  $T$  has exactly two proper ideal, namely,  $0$  and  $K\alpha$ , hence we have two cases, either  $(R : T) = 0$  or  $(R : T) = K\alpha$ . If  $(R : T) = K\alpha$ , then one can easily see that  $R = S + K\alpha$ , for some  $S \in \text{RgMax}(K)$ . Therefore  $R$  satisfies in condition (1) and we are done. Thus assume that  $(R : T) = 0$  which clearly is not a prime ideal of  $T$ . Hence by the above comments we infer that  $T$  is integral over  $R$ , i.e.,  $0 = (R : T) \in \text{Max}(R)$  which means that  $R$  is a field. Now, since  $T$  is a non-field local minimal ring extension of the field  $R$ , by the above comments ([21, Lemma 1.2]) we deduce that  $T \cong R[y]/(y^2)$  and therefore we conclude that  $R \cong K$ . Assume that  $f : K \rightarrow R$  be a ring isomorphism. Hence for each  $x \in K$ , there exist unique elements  $\sigma(x)$  and  $\delta(x)$  in  $K$  such that  $f(x) = \sigma(x) + \delta(x)\alpha$ . Since  $f$  is a ring isomorphism one can easily see that  $\sigma$  is a ring endomorphism of  $K$  and  $\delta$  is a  $\sigma$ -derivation of  $K$ . Clearly,  $\sigma$  is one-one. Finally, note that since  $R$  is a field and  $\alpha^2 = 0$ , we deduce that  $\alpha \notin R$ , which by maximality of  $R$  we conclude that  $K \oplus K\alpha = T = R[\alpha] = R \oplus R\alpha = \{\sigma(x) + \delta(x)\alpha \mid x \in K\} \oplus \{\sigma(z)\alpha \mid z \in K\}$ . The latter equality immediately implies that  $\sigma$  is onto and therefore  $\sigma$  is a field automorphism of  $K$ . Hence  $R$  satisfies in condition (2) and we are done.  $\square$

Now assume that  $K$  is a field,  $\sigma \in \text{Aut}(K)$  and  $\delta$  is a  $\sigma$ -derivation of  $K$ . If  $F$  is the prime subfield of  $K$ , then one can easily see that for each  $x \in F$ , we have  $\delta(x) = 0$  (note,  $\delta(1) = 0$ ). Moreover, if  $x \in K$  is algebraic over  $F$ , then it is not hard to see that  $\delta(x) = 0$ . Thus if  $K$  is algebraic over its prime subfield then the only  $\sigma$ -derivation of  $K$  is  $0$ . Now the following immediate corollaries are in order.

**Corollary 3.12.** *Let  $K$  be a field which is algebraic over its prime subfield and  $T = K[\alpha]$ , where  $\alpha^2 = 0$ . Then  $R$  is a maximal subring of  $T$  if and only if either  $R = K$  or  $R = S + K\alpha$  where  $S \in \text{RgMax}(K)$ .*

**Corollary 3.13.** *Let  $K$  be a field and  $T = K[\alpha]$ , where  $\alpha^2 = 0$ . Then  $T$  has finitely many maximal subrings if and only if  $K$  has only finitely many maximal subrings. Moreover in this case we have  $|\text{RgMax}(T)| = 1 + |\text{RgMax}(K)|$ .*

**Remark 3.14.** Let  $R \subseteq T$  be an extension of rings and  $X$  be a minimal generating set for  $T$  as a ring over  $R$ . Then  $|\text{RgMax}(T)| \geq |X|$ . To see this, assume that for each  $x \in X$ , let  $A_x = X \setminus \{x\}$  and  $S_x = R[A_x]$ . Hence we infer that for each  $x \in X$ ,  $S_x$  is a proper subring of  $T$  and  $S_x[x] = T$ . Therefore by [2, Theorem 2.5],  $T$  has a maximal subring  $T_x$  which contains  $S_x$  (therefore,  $R, A_x \subseteq T_x$ ) and  $x \notin T_x$ . This immediately shows that  $|\text{RgMax}(T)| \geq |X|$ .

**Lemma 3.15.** *Let  $R$  be a ring and  $D$  be a UFD subring of  $R$ . If  $|U(R) \cap \text{Irr}(D)| \geq n$ , then there exists a chain  $R_n \subset R_{n-1} \subset \cdots \subset R_1 \subset R_0 = R$ , where each  $R_i$  is a maximal subring of  $R_{i-1}$ , for  $1 \leq i \leq n$ . In particular, if  $|U(R) \cap \text{Irr}(D)|$  is infinite, then there exists an infinite descending chain  $\cdots \subset R_1 \subset R_0 = R$ , where each  $R_i$  is a maximal subring of  $R_{i-1}$ , for  $i \geq 1$ .*

*Proof.* Note that by the proof of [11, Theorem 1.3], if  $p \in U(R) \cap \text{Irr}(D)$ , then  $R$  has a maximal subring  $R_1$  such that  $D \subseteq R_1$  and  $\text{Irr}(D) \setminus \{p\} \subseteq U(R_1)$ . Hence by repeating this process we can find the desired descending chain.  $\square$

Finally, we have the following generalization of (2) of Corollary 2.1.

**Theorem 3.16.** *Let  $R$  be an uncountable PID, then  $|\text{RgMax}(R)| \geq |R|$ . Moreover, there exists an infinite descending chain  $\cdots \subset R_1 \subset R_0 = R$ , where each  $R_i$  is a maximal subring of  $R_{i-1}$ , for  $i \geq 1$ .*

*Proof.* We have two cases. (a) If  $|U(R)| = |R|$ , then by (1) of [11, Proposition 1.13], we infer that  $|\text{RgMax}(R)| \geq |R|$ . Also note that there exists an algebraically independent set  $X \subseteq U(R)$  over the prime subring  $Z$  of  $R$  such that  $|X| = |U(R)| = |R|$ . Thus by Lemma 3.15,  $R$  has an infinite descending chain of maximal subrings.

(b) If  $|U(R)| < |R|$ , then since  $R$  is a PID, and therefore is an atomic domain, we infer that  $|\text{Irr}(R)| = |R|$ . Now we have two cases:

(b1) If  $R$  has zero characteristic, then note that since  $R$  is a PID (and therefore is an UFD), only for countably many  $p \in \text{Irr}(R)$  we have  $\mathbb{Z} \cap Rp \neq 0$ , for otherwise since  $\mathbb{Z}$  has countably many ideals, we conclude that there exists an element  $n \neq 0$  in  $\mathbb{Z}$  such that  $n$  has uncountable many non associate irreducible divisor in  $R$ , which is a contradiction. Thus there exist  $A \subseteq \text{Irr}(R)$  such that  $\text{Irr}(R) \setminus A$  is countable and for each  $q \in A$  we have  $\mathbb{Z} \cap Rq = 0$ . Thus  $|A| = |R|$  and for each  $q \in A$ ,  $R/Rq$  is a field with zero characteristic. Thus  $R/Rq$  is submaximal by (1) of Corollary 2.1. Hence we infer that  $R$  has a maximal subring, say  $S_q$ , such that  $Rq \subseteq S_q$ . It is clear that whenever  $q \neq q'$  are in  $A$ , then  $S_q \neq S_{q'}$  for  $Rq + Rq' = R$ . Thus we infer that  $|\text{RgMax}(R)| \geq |A| = |R|$ .

(b2) Now assume that  $R$  has nonzero characteristic and  $q \in \text{Irr}(R)$ . It is clear that  $q$  is not algebraic over  $Z$ , the prime subring of  $R$ . Let  $\{A_i\}_{i \in I}$  be a partition of  $\text{Irr}(R) \setminus \{q\}$ , such that  $|A_i| = |I| = |\text{Irr}(R)| = |R|$ . For each  $i \in I$ , and  $q' \in A_i$ ,  $q + (q')$  is a unit in  $R/Rq'$ . Hence if for each  $q' \in A_i$ ,  $q + (q')$  is algebraic over the prime subring of  $R/Rq'$ , then we infer that  $Z[q] \cap Rq' \neq 0$ . Since  $A_i$  is uncountable and  $Z[q]$  is countable, we infer that there exists a nonzero element  $f \in Z[q]$  such that  $f$  is divisible by uncountably many elements of  $A_i$ , which is a contradiction, for  $R$  is a PID. Thus for each  $i \in I$ , there exists  $q_i \in A_i$  such that the field  $R/Rq_i$  is not algebraic over its prime subfield. Thus, by (3) of Corollary 2.1,  $R/Rq_i$  has a maximal subring,  $S_i/Rq_i$ , where  $S_i$  is a subring of  $R$ . It is clear that  $S_i$  is a maximal subring of  $R$  and by a similar proof of the previous case whenever  $i \neq j$  are in  $I$ , we have  $S_i \neq S_j$ . Thus  $|\text{RgMax}(R)| \geq |I| = |R|$ .

Finally, in case (b) we infer that  $R$  has a maximal ideal  $M$  such that  $R/M$  is not an absolutely algebraic field. Hence in this case by Corollary 2.5, the infinite descending chain of maximal subrings exists for  $R$ , too.  $\square$

## REFERENCES

- [1] D.D. Anderson, D.E. Dobbs and B. Mullins, The primitive element theorem for commutative algebra, Houston J. Math., **25** (1999), 603-623. Corrigendum, Huston J. Math. **28** (2002) 217-219.

- [2] A. Azarang, On maximal subrings, *Far East J. Math. Sci. (FJMS)* **32** (1) (2009) 107-118.
- [3] A. Azarang, Submaximal integral domains, *Taiwanese J. Math.*, **17** (4) (2013) 1395-1412.
- [4] A. Azarang, The space of maximal subrings of a commutative ring, *Comm. Alg.* **43** (2) (2015) 795-811.
- [5] A. Azarang, On the existence of maximal subrings in commutative noetherian rings, *J. Algebra Appl.* **14** (1) (2015) ID:1450073.
- [6] A. Azarang, O.A.S. Karamzadeh, On the existence of maximal subrings in commutative artinian rings, *J. Algebra Appl.* **9** (5) (2010) 771-778.
- [7] A. Azarang, O.A.S. Karamzadeh, Which fields have no maximal subrings?, *Rend. Sem. Mat. Univ. Padova*, **126** (2011) 213-228.
- [8] A. Azarang, O.A.S. Karamzadeh, On Maximal Subrings of Commutative Rings, *Algebra Colloquium*, **19** (Spec 1) (2012) 1125-1138.
- [9] A. Azarang, O.A.S. Karamzadeh, Most Commutative Rings Have Maximal Subrings, *Algebra Colloquium*, **19** (Spec 1) (2012) 1139-1154.
- [10] A. Azarang, O.A.S. Karamzadeh, A. Namazi, Hereditary properties between a ring and its maximal subrings, *Ukrainian. Math. J.* **65** (7) (2013) 981-994.
- [11] A. Azarang, G. Oman, Commutative rings with infinitely many maximal subrings, *J. Algebra Appl.* **13** (7) (2014) ID:1450037.
- [12] H.E. Bell and A.A. Klein, On finiteness of rings with finite maximal subrings. *J. Math. Math. Sci.* **16** (2) (1993) 351-354.
- [13] H.E. Bell, Rings with finitely many subrings, *Math. Ann.* (1969) 314-318.
- [14] J.V. Brawley and G.E. Schnibben, Infinite algebraic extensions of finite fields, *Contemporary mathematics*, 1989.
- [15] P.M. Cohn, *Basic Algebra (Groups, Rings and Fields)*, Springer-Verlag, London-Berlin-Heidelberg, 2002.
- [16] L.I. Dechene, *Adjacent Extensions of Rings*, Ph.D. Dissertation, University of California, Riverside, (1978).
- [17] D.E. Dobbs, B. Mullins, and M. Picavet-l'Hermitte, The singly generated unital rings with only finitely many unital subrings, *Comm. Algebra* **36** (2008) 2638-2653.
- [18] D.E. Dobbs, B. Mullins, G. Picavet, and M. Picavet-l'Hermitte, On the *FIP* property for extensions of commutative rings, *Comm. Algebra* **33** (2005) 3091-3119.
- [19] D.E. Dobbs, G. Picavet, and M. Picavet-l'Hermitte, A Characterization of the commutative unital rings with only finitely many unital subrings, *J. Algebra Appl.* **7** (5) (2008) 601-622.
- [20] D.E. Dobbs, G. Picavet, and M. Picavet-l'Hermitte, Characterizing the ring extensions that satisfy *FIP* or *FCP*, *J. Algebra* **371** (2012) 391-429.
- [21] D. Ferrand, J.-P. Olivier, Homomorphismes minimaux danneaux, *J. Algebra* **16** (1970) 461-471.
- [22] R. Gilmer, A note on rings with only finitely many subrings, *Scripta Math.* **29** (1973) 37-38.
- [23] R. Gilmer, Some finiteness conditions on the set of overring an integral domain. *Proc. A.M.S.* **131** (2003) 2337-2346.
- [24] I. Kaplansky, *Commutative Rings*, rev. ed. University of Chicago Press, Chicago, 1974.
- [25] A.A. Klein, The finiteness of a ring with a finite maximal subrings. *Comm. Algebra* **21** (4) (1993) 1389-1392.
- [26] S.S. Korobkov, Finite Rings with Exactly Two Maximal Subrings, *Russian Mathematics (Iz. VUZ)*, **55** (6) (2011) 4652.
- [27] T.J. Laffey, A finiteness theorem for rings, *Proc. R. Ir. Acad.* **92** (2) (1992) 285-288.
- [28] T. Kwen Lee, K. Shan Liu, Algebra with a finite-dimensional maximal subalgebra. *Comm. Algebra* **33** (1) (2005) 339-342.
- [29] M.L. Modica, *Maximal Subrings*, Ph.D. Dissertation, University of Chicago, 1975.
- [30] S. Roman, *Field Theory*, Sec. ed. Springer-Verlag, 2006.
- [31] A. Rosenfeld, A note on two special types of rings, *Scripta Math.* **28** (1967) 51-54.
- [32] T. Szele, On a finiteness criterion for modules. *Publ. Math. Debrecen.* **3** (1954) 253-256.
- [33] Oscar Zariski, A new proof of Hilbert's Nullstellensatz, *Bull. Amer. Math. Soc.*, **53** (1947) 362-368.